Tech report: the affine Particle-in-Cell method

Chenfanfu Jiang†  Craig Schroeder†  Andrew Selle*  Joseph Teran†  Alexey Stomakhin⋆
†University of California Los Angeles   *Walt Disney Animation Studios

1 Introduction


2 Notation

We begin by laying out the notation that we use in this document. Many quantities have subscripts ($m_p$), and some have superscripts as well ($v^n_p$). Subscripts $p$ and $q$ are used to refer to particles, and subscripts $i$ and $j$ are used to refer to regular grid indices. The superscript distinguishes quantities available or computed at the beginning of the time step ($v^n_p$) from quantities that are computed for use at the beginning of the next time step ($v^{n+1}_p$). We use lowercase bold for vectors ($v^n_p$) and uppercase bold for matrices ($B^n_p$), with the exception of angular momentum, a vector quantity that is normally denoted by $L^n_p$. The notation we use is summarized in Table 1.

3 Preliminaries

When we consider conservation of angular momentum when transferring from grid to particles at the end of a time step, we need to consider angular momentum to be defined over moved grid nodes and we use the notation $\tilde{x}^{n+1}_i = x_i + \Delta t \tilde{v}^{n+1}_i$ to indicate this. To avoid confusion, rather than referring to unmoved grid nodes at the beginning of the time step as $x_i$, we will use $x^n_i$ to emphasize that they have not been dynamically updated yet, whereas the $\tilde{x}^{n+1}_i$ have been. We adopt this notation in the remainder of the document.

We will also use a few properties of standard interpolating functions, namely.

$$\sum_i w^n_{ip} = 1$$
$$\sum_i w^n_{ip} x^n_i = x^n_p$$
$$\sum_i w^n_{ip} (x^n_i - x^n_p) = 0$$

Here $w^n_{ip} = N_i(x^n_p)$ are the weights at time $t^n$, where $N_i(x)$ is the interpolation function associated with grid node $i$.

The definitions of linear and angular momentum on the grid as well as linear momentum on particles are listed below. The angular momentum on particles is written in different forms for RPIC and APIC and will be defined in the corresponding sections.

**Definition 3.1.** The total linear momentum on the grid (after the particle-to-grid transfer at time $n$) is

$$p^{G,n}_{tot} = \sum_i m^n_i v^n_i$$

**Definition 3.2.** The total angular momentum on the grid (after the particle-to-grid transfer at time $n$) is

$$L^{G,n}_{tot} = \sum_i x^n_i \times m^n_i v^n_i$$

**Definition 3.3.** The total linear momentum on the grid (after the grid dynamics, before the grid-to-particle transfer in the end of time $n$) is

$$p^{G,n+1}_{tot} = \sum_i m^n_i \tilde{v}^{n+1}_i$$

**Definition 3.4.** The total angular momentum on the grid (after the grid dynamics, before the grid-to-particle transfer in the end of time $n$) is

$$L^{G,n+1}_{tot} = \sum_i \tilde{x}^{n+1}_i \times m^n_i \tilde{v}^{n+1}_i$$

**Definition 3.5.** The total linear momentum on particles (before the particle-to-grid transfer at time $n$) is

$$p^{P,n}_{tot} = \sum_p m_p v^n_p$$
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Location</th>
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<tbody>
<tr>
<td>(m_p)</td>
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<tr>
<td>(x_p^n)</td>
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<tr>
<td>(v_p^n)</td>
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<td>(L_p^n)</td>
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<td>(x_i^n)</td>
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<td>(\tilde{x}_{i+1}^n)</td>
<td>n</td>
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<tr>
<td>(v_i^n)</td>
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<td>(\tilde{v}_{i+1}^n)</td>
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<td>(w_{n+p})</td>
<td>n+p</td>
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<tr>
<td>(\nabla w_{n+p})</td>
<td>n+p</td>
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<tr>
<td>(P_{tot}^{p,n})</td>
<td>g</td>
<td>v</td>
<td>total particle linear momentum</td>
</tr>
<tr>
<td>(P_{tot}^{g,n})</td>
<td>g</td>
<td>v</td>
<td>total grid linear momentum</td>
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<tr>
<td>(L_{tot}^{p,n})</td>
<td>g</td>
<td>v</td>
<td>total particle angular momentum</td>
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<td>(L_{tot}^{g,n})</td>
<td>g</td>
<td>v</td>
<td>total grid angular momentum</td>
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<td>(\omega^\epsilon)</td>
<td>g</td>
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<td>cross product matrix of (v)</td>
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<tr>
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<tr>
<td>(\Delta t)</td>
<td>g</td>
<td>s</td>
<td>time step size</td>
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</tbody>
</table>

Table 1: Summary of notation used in this paper. Locations are p (particle), n (regular grid node), or g (global; does not live at any location in space). Quantities are of type s (scalar), v (vector), m (matrix) or t (rank-3 tensor).

**Definition 3.6.** The total linear momentum on particles (after the grid-to-particle transfer in the end of time \(n\)) is
\[
P_{tot}^{p,n+1} = \sum_p m_p v_p^{n+1}
\]

**4 Piecewise rigid**

Here is a data flow diagram for Rigid Particle-in-Cell method.

RPIC stores mass \(m_p\), position \(x_p\), velocity \(v_p\) and angular momentum \(L_p\) on particles. The transfer from particles to the grid are given by

\[
m_i^n = \sum_p w_{ip}^n m_p
\]
\[
K_p^n = \sum_j w_{jp} m_p (x_j^n - x_p^n)^T (x_j^n - x_p^n)
\]
\[
m_i^n v_i^n = \sum_p w_{ip}^n m_p (v_p^n + ((K_p^n)^{-1} L_p^n) \times (x_i^n - x_p^n)).
\](1)

One may imagine this transfer as distributing the masses \(w_{ip}^n m_p\) from a rigid body to the grid node \(i\). \(K_p^n\) is the inertia tensor associated with the local rigid body represented by particle \(p\). For any vectors \(u\) and \(v\), \(u^\times\) is defined to be the cross product matrix of \(u\), so that \(u^\times v = u \times v\) and \((u^\times)^T v = v \times u\).

The transfer from the grid to particles are given by

\[
v_{p+1}^n = \sum_i w_{ip} \tilde{v}_{i+1}^n
\]
\[
L_{p+1}^{n+1} = \sum_i w_{ip} (x_i^n - x_p^n) \times m_p \tilde{v}_{i+1}^n
\](2)
4.1 Preservation of rigid motion velocity field

**Proposition 4.1.** Let $\Delta t = 0$ and consider the process of transferring grid velocity $\tilde{v}_i^{n+1}$ to particles $(u_p^{n+1}, L_p^{n+1})$ and then back to the grid $v_i^{n+1}$ with the Rigid Particle-in-Cell method. If the velocities before the transfer represent rigid motion, $\tilde{v}_i^{n+1} = v + \omega \times \mathbf{x}_i^n$, where $v$ and $\omega$ are any constant vectors, then after the process, this velocity field is exactly reproduced: $u_p^{n+1} = \tilde{v}_i^{n+1}$.

**Proof.** Since $\Delta t = 0$, we have $u_p^{n+1} = w_{ip}^{n+1}$ and $\mathbf{x}_p^n = \mathbf{x}_p^{n+1}$, so that $K_p^n = K_p^{n+1}$ and $m_p^n = m_p^{n+1}$. The grid to particle transfer is (for $v_p$)

$$v_p^{n+1} = \sum_i w_{ip}^n (v + \omega \times \mathbf{x}_i^n)$$

$$= \sum_i w_{ip}^n (v + \omega \times \mathbf{x}_i^n)$$

$$= v + \omega \times \sum_i w_{ip}^n \mathbf{x}_i^n$$

Therefore, the rigid motion velocity field represented by $v_p$ and $\omega$ is preserved in the full transfer cycle.

A particle to grid transfer is then performed:

$$m_i^{n+1} v_i^{n+1} = \sum_p m_p u_p^{n+1} + ((K_p^{n+1})^{-1} L_p^{n+1}) \times (\mathbf{x}_i^n - \mathbf{x}_p^n)$$

$$= \sum_i m_i v_i^n + \omega$$

$$= \mathbf{x}_i^{n+1}$$

We can see $v_i^{n+1} = v + \omega \times \mathbf{x}_i^n = \tilde{v}_i^{n+1}$. Therefore, the rigid motion velocity field represented by $v$ and $\omega$ is preserved in the full transfer cycle.

4.2 Conservation of linear momentum

4.2.1 Particle to grid

**Proposition 4.2.** Linear momentum is conserved during the RPIC transfer from particles to the grid. $p_{tot}^{G,n} = p_{tot}^{F,n}$ under transfer $1$. 
The linear momentum on the grid after transferring from particles is
\[
p_{G,n}^{\text{tot}} = \sum_i m_i^n v_i^n
\]
\[
= \sum_i \left( \sum_p w^n_{ip} m_p v_{p}^{n} + \left( \left( \mathbf{K}_p^n \right)^{-1} \mathbf{L}_p^n \right) \times (x_{i}^{n} - x_{p}^{n}) \right)
\]
\[
= \sum_{i,p} w^n_{ip} m_p v_{p}^{n} + \sum_{i,p} w^n_{ip} m_p \left( \left( \mathbf{K}_p^n \right)^{-1} \mathbf{L}_p^n \right) \times (x_{i}^{n} - x_{p}^{n})
\]
\[
= \sum_{p} \left( \sum_{i} w^n_{ip} \right) m_p v_{p}^{n} + \sum_{p} m_p \left( \left( \mathbf{K}_p^n \right)^{-1} \mathbf{L}_p^n \right) \times \left( \sum_{i} w^n_{ip} (x_{i}^{n} - x_{p}^{n}) \right)
\]
\[
= \sum_{p} m_p v_{p}^{n}
\]
\[
= p_{G,n}^{\text{tot}}
\]

4.2.2 Grid to particle

Proposition 4.3. Linear momentum is conserved during the RPIC transfer from the grid to particles. \(p_{P,n}^{n+1} = p_{G,n}^{n+1}\) under transfer 2.

Proof. The linear momentum on the particles after transferring from the grid is
\[
p_{P,n}^{n+1} = \sum_p m_p v_{p}^{n+1}
\]
\[
= \sum_p \left( \sum_{i} w^n_{ip} \right) m_p v_{p}^{n+1}
\]
\[
= \sum_{i} \left( \sum_{p} w^n_{ip} m_p \right) v_{p}^{n+1}
\]
\[
= \sum_{i} m_i^n v_{i}^{n+1}
\]
\[
= p_{P,n}^{n+1}
\]

4.3 Conservation of angular momentum

With \(m_p, x_p, v_p\) and \(L_p\), we can define the total angular momentum on particles in the piecewise rigid case.

Definition 4.1. The total angular momentum on particles (before the particle-to-grid transfer at time \(n\)) represented by RPIC is
\[
L_{P,n}^{\text{tot}} = \sum_p (x_{p}^{n} \times m_p v_{p}^{n} + L_p^n)
\]

Definition 4.2. The total angular momentum on particles (after the grid-to-particle transfer in the end of time \(n\)) represented by RPIC is
\[
L_{P,n+1}^{\text{tot}} = \sum_p (x_{p}^{n+1} \times m_p v_{p}^{n+1} + L_p^{n+1})
\]

4.3.1 Particle to grid

Proposition 4.4. Angular momentum is conserved during the RPIC transfer from particles to the grid. \(L_{G,n}^{\text{tot}} = L_{P,n}^{\text{tot}}\) under transfer 1.
Proof. The angular momentum on the particles after transferring from the grid is

\[ L_{tot}^{G,n} = \sum_i x_i^n \times m_i^n v_i^n \]

\[ = \sum_{i,p} x_i^n \times (\sum_p w_{ip}^n w_p m_p (\mathbf{K}_p^{n-1} L_p^{n-1}) \times (x_i^n - x_p^n)) \]

\[ = \sum_{i,p} x_i^n \times w_{ip}^n w_p m_p v_p^n + \sum_{i,p} x_i^n \times w_{ip}^n m_p ((\mathbf{K}_p^{n-1} L_p^{n-1}) \times (x_i^n - x_p^n)) \]

\[ = \sum_{i,p} \left( \sum_i w_{ip}^n x_i^n \right) \times m_p v_p^n + \sum_{i,p} x_i^n \times w_{ip}^n m_p (x_i^n - x_p^n)^T (\mathbf{K}_p^{n-1}) L_p^{n-1} \]

\[ = \sum_{i,p} x_i^n \times m_p v_p^n + \sum_{i,p} (\sum_i m_p w_{ip}^n (x_i^n - x_p^n) (x_i^n - x_p^n)^T) (\mathbf{K}_p^{n-1}) L_p^{n-1} + \sum_{i,p} x_i^n \times \left( \sum_i w_{ip}^n (x_i^n - x_p^n)^T (\mathbf{K}_p^{n-1}) L_p^{n-1} \right) \]

\[ = \sum_{i,p} x_i^n \times m_p v_p^n + \sum_{i,p} \mathbf{K}_p^{n-1} L_p^{n-1} \]

\[ = \sum_{i,p} (x_i^n \times m_p v_p^n + L_p^{n-1}) \]

\[ = L_{tot}^{G,n} \]

4.3.2 Grid to particle

Proposition 4.5. Angular momentum is conserved during the RPIC transfer from the grid to particles. \( L_{tot}^{P,n+1} = L_{tot}^{G,n+1} \) under transfer 2.

Proof. The angular momentum on the particles after transferring from the grid is

\[ L_{tot}^{P,n+1} = \sum_i \left( x_i^{n+1} \times m_i v_i^{n+1} + L_i^{n+1} \right) \]

\[ = \sum_{i,p} \left( x_i^{n+1} \times m_p \left( \sum_i w_{ip}^n v_i^{n+1} + w_{ip}^n (x_i^n - x_p^n) \times m_p v_i^{n+1} \right) \right) \]

\[ = \sum_{i,p} \left( x_i^{n+1} \times m_p w_{ip}^n v_i^{n+1} + w_{ip}^n (x_i^n - x_p^n) \times m_p v_i^{n+1} \right) \]

\[ = \sum_{i,p} \left( x_i^{n+1} - x_p^n \right) \times m_p w_{ip}^n v_i^{n+1} + \sum_{i,p} w_{ip}^n \sum_i w_{ip}^n v_i^{n+1} \times m_p v_i^{n+1} \]

\[ = \Delta t \sum_{i,p} v_i^{n+1} \times m_p \sum_i w_{ip}^n \sum_i v_i^{n+1} + \sum_{i,p} w_{ip}^n (\tilde{x}_i^{n+1} - \Delta t v_i^{n+1}) \times m_p \tilde{v}_i^{n+1} \]

\[ = \Delta t \sum_{i,p} v_i^{n+1} \times m_p v_i^{n+1} + \sum_{i,p} w_{ip}^n \tilde{x}_i^{n+1} \times m_p \tilde{v}_i^{n+1} \]

\[ = \sum_{i,p} \left( \sum_i w_{ip}^n \right) \tilde{x}_i^{n+1} \times \tilde{v}_i^{n+1} \]

\[ = \sum_{i,p} \tilde{x}_i^{n+1} \times \tilde{v}_i^{n+1} \]

\[ = L_{tot}^{G,n+1} \]

5 Piecewise Affine

Here is a data flow diagram for Affine Particle-in-Cell method.
APIC stores mass $m_p$, position $x_p$, velocity $v_p$, and matrix $B_p$ on particles. The transfer from particles to the grid are given by

$$
\begin{align*}
    m_i^n &= \sum_p w_{ip}^n m_p \\
    D_p^n &= \sum_i w_{ip}^n (x_i^n - x_p^n) (x_i^n - x_p^n)^T = \sum_i w_{ip}^n x_i^n (x_i^n)^T - x_p^n (x_p^n)^T \\
    m_i^n v_i^n &= \sum_p w_{ip}^n m_p (v_p^n + B_p^n (D_p^n)^{-1} (x_i^n - x_p^n)) 
\end{align*}
$$

with the transfer to particles given by

$$
\begin{align*}
    v_p^{n+1} &= \sum_i w_{ip}^n v_i^{n+1} \\
    B_p^{n+1} &= \sum_i w_{ip}^n v_i^{n+1} (x_i^n - x_p^n)^T. 
\end{align*}
$$

### 5.1 Preservation of affine velocity fields

**Proposition 5.1.** Let $\Delta t = 0$ and consider the the process of transferring velocity $(\tilde{v}_i^{n+1})$ information to particles $(v_p^{n+1}, B_p^{n+1})$ and then back to the grid $(\tilde{v}_i^{n+1})$ with the Affine Particle-in-Cell method. If the velocities before the transfer represent an affine velocity field, $\tilde{v}_i^{n+1} = v + C x_i^n$, where $v$ is a vector and $C$ is a matrix, then after the process, this velocity field is exactly reproduced: $v_p^{n+1} = \tilde{v}_i^{n+1}$.

**Proof.** Since $\Delta t = 0$, we have $w_{ip}^n = w_{ip}^{n+1}$ and $x_p^n = x_p^{n+1}$, so that $D_p^n = D_p^{n+1}$ and $m_p^n = m_p^{n+1}$. The grid to particle transfer is (for $v_p$)

$$
\begin{align*}
    v_p^{n+1} &= \sum_i w_{ip}^n v_i^{n+1} \\
    &= \sum_i w_{ip}^n (v + C x_i^n) \\
    &= \sum_i w_{ip}^n v + \sum_i w_{ip}^n C x_i^n \\
    &= v + \sum_i w_{ip}^n C \sum_i w_{ip}^n x_i^n \\
    &= v + C x_p^n 
\end{align*}
$$

and for $B_p$:

$$
\begin{align*}
    B_p^{n+1} &= \sum_i w_{ip}^n v_i^{n+1} (x_i^n - x_p^n)^T \\
    &= \sum_i w_{ip}^n (v + C x_i^n) (x_i^n - x_p^n)^T \\
    &= \sum_i w_{ip}^n v (x_i^n - x_p^n)^T + \sum_i w_{ip}^n C x_i^n (x_i^n - x_p^n)^T \\
    &= v \left( \sum_i w_{ip}^n (v (x_i^n - x_p^n)^T) \right) + C \sum_i w_{ip}^n (x_i^n - x_p^n) (x_i^n - x_p^n)^T + \sum_i w_{ip}^n C x_i^n (x_i^n - x_p^n)^T \\
    &= C D_p^n + C x_p^n \left( \sum_i w_{ip}^n (x_i^n - x_p^n) \right)^T \\
    &= C D_p^n 
\end{align*}
$$

A particle to grid transfer is then performed:

$$
\begin{align*}
    m_i^{n+1} v_i^{n+1} &= \sum_p w_{ip}^{n+1} m_p (v_p^{n+1} + B_p^{n+1} (D_p^n)^{-1} (x_i^{n+1} - x_p^{n+1})) \\
    m_i^n v_i^n &= \sum_p w_{ip}^n m_p (v_p^{n+1} + B_p^{n+1} (D_p^n)^{-1} (x_i^n - x_p^n)) \\
    &= \sum_p w_{ip}^n m_p (v + C x_i^n + C D_p^n (D_p^n)^{-1} (x_i^n - x_p^n)) \\
    &= \sum_p w_{ip}^n m_p (v + C x_i^n + C (x_i^n - x_p^n)) \\
    &= \sum_p w_{ip}^n m_p (v + C x_i^n) \\
    &= m_i^n (v + C x_i^n) 
\end{align*}
$$

We can see $v_p^{n+1} = v + C x_p^n = \tilde{v}_i^{n+1}$. Therefore, the affine velocity field represented by $v$ and $C$ is preserved in the full transfer cycle.
5.2 Conservation of linear momentum

5.2.1 Particle to grid

Proposition 5.2. Linear momentum is conserved during the APIC transfer from particles to the grid. \( \mathbf{p}^{G,n}_{tot} = \mathbf{p}^{P,n}_{tot} \) under transfer 3.

Proof. The linear momentum on the grid after transferring from particles is

\[
\mathbf{p}^{G,n}_{tot} = \sum_{i} m_i v^n_i = \sum_{p} \sum_{i} m_p w^n_{ip} (v^n_i + B^n_p (D^n_p)^{-1} (x^n_i - x^n_i)) \\
= \sum_{p} m_p v^n_p + \sum_{p} m_p w^n_{ip} B^n_p (D^n_p)^{-1} (x^n_i - x^n_p) \\
= \sum_{p} m_p \left( \sum_{i} w^n_{ip} \right) v^n_p + \sum_{p} m_p B^n_p (D^n_p)^{-1} \left( \sum_{i} w^n_{ip} (x^n_i - x^n_p) \right) \\
= \sum_{p} m_p v^n_p = \mathbf{p}^{P,n}_{tot}
\]

5.2.2 Grid to particle

Proposition 5.3. Linear momentum is conserved during the APIC transfer from the grid to particles. \( \mathbf{p}^{P,n+1}_{tot} = \mathbf{p}^{G,n+1}_{tot} \) under transfer 4.

Proof. The linear momentum on the particles after transferring from the grid is

\[
\mathbf{p}^{P,n+1}_{tot} = \sum_{p} m_p v^{n+1}_p = \sum_{p} m_p \left( \sum_{i} w^{n+1}_{ip} \right) v^{n+1}_i \\
= \sum_{i} m^n_i \tilde{v}^{n+1}_i = \mathbf{p}^{G,n+1}_{tot}
\]

5.3 Conservation of angular momentum

With \( m_p, x_p, v_p \) and \( B_p \), we can define the total angular momentum on particles in the piecewise affine case.

Definition 5.1. The total angular momentum on particles (before the particle-to-grid transfer at time \( n \)) represented by APIC is

\[
\mathbf{L}^{P,n}_{tot} = \sum_{p} x^n_p \times m_p v^n_p + \sum_{p} m_p (B^n_p) : \epsilon
\]

Definition 5.2. The total angular momentum on particles (after the grid-to-particle transfer in the end of time \( n \)) represented by APIC is

\[
\mathbf{L}^{P,n+1}_{tot} = \sum_{p} x^{n+1}_p \times m_p v^{n+1}_p + \sum_{p} m_p (B^{n+1}_p) : \epsilon
\]

Note we use the permutation tensor \( \epsilon \) in this section. To make these portions easier to read, we take the convention that for any matrix \( A \), the contraction \( A : \epsilon \) means the same thing as \( A_{\alpha \beta} \epsilon^{\alpha \beta \gamma} \). The manipulation \( \mathbf{u} \times \mathbf{v} = (\mathbf{v} \mathbf{u}^T) : \epsilon \) is used to transition from a cross product into the permutation tensor.
5.3.1 Particle to grid

**Proposition 5.4.** Angular momentum is conserved during the APIC transfer from particles to the grid. \( L_{\text{tot}}^{G,n} = L_{\text{tot}}^{P,n} \) under transfer 3.

**Proof.** The angular momentum on the grid after transferring from particles is

\[
L_{\text{tot}}^{G,n} = \sum_i x_i^n \times m_i u_i^n
\]

\[
= \sum_p \sum_i x_i^n \times m_p u_{ip}^n (v_p^i + B_p^n(D_p^n)^{-1}(x_i^n - x_p^n))
\]

\[
= \sum_p \sum_i x_i^n \times m_p u_{ip}^n u_p^n + \sum_p \sum_i x_i^n \times m_p u_{ip}^n B_p^n(D_p^n)^{-1} x_i^n - \sum_p \left( \sum_i w_{ip}^n x_i^n \right) \times m_p B_p^n(D_p^n)^{-1} x_p^n
\]

\[
= \sum_p x_p^n \times m_p v_p^n + \sum_p \sum_i w_{ip}^n x_i^n \times (B_p^n(D_p^n)^{-1} x_i^n) - \sum_p m_p x_p^n \times (B_p^n(D_p^n)^{-1} x_p^n)
\]

\[
= \sum_p x_p^n \times m_p v_p^n + \sum_p \left( \sum_i w_{ip}^n B_p^n(D_p^n)^{-1} x_i^n (x_p^n)^T \right) : \epsilon - \sum_p B_p^n(D_p^n)^{-1} x_p^n (v_p^n)^T : \epsilon
\]

\[
= \sum_p x_p^n \times m_p v_p^n + \sum_p \left( \sum_i w_{ip}^n B_p^n(D_p^n)^{-1} x_i^n (x_p^n)^T - B_p^n(D_p^n)^{-1} x_p^n (v_p^n)^T \right) : \epsilon
\]

\[
= \sum_p x_p^n \times m_p v_p^n + \sum_p \left( \sum_i w_{ip}^n B_p^n(D_p^n)^{-1} (x_i^n)^T - (x_p^n)^T \right) : \epsilon
\]

\[
= \sum_p x_p^n \times m_p v_p^n + \sum_p (B_p^n(D_p^n)^{-1})^T : \epsilon
\]

\[
= \sum_p x_p^n \times m_p v_p^n + \sum_p (B_p^n(D_p^n)^{-1} D_p^n)^T : \epsilon
\]

\[
= L_{\text{tot}}^{P,n}
\]

5.3.2 Grid to particle

**Proposition 5.5.** Angular momentum is conserved during the APIC transfer from the grid to particles. \( L_{\text{tot}}^{P,n+1} = L_{\text{tot}}^{G,n+1} \) under transfer 4.

**Proof.** As before, the manipulation \((v u^T)^T : \epsilon = u \times v\) is used to convert the permutation tensor into a cross product. The angular momentum on the particles after transferring from the grid is

\[
L_{\text{tot}}^{P,n+1} = \sum_p x_p^{n+1} \times m_p v_p^{n+1} + \sum_p m_p(B_p^{n+1})^T : \epsilon
\]

\[
= \sum_p x_p^{n+1} \times m_p v_p^{n+1} + \sum_p \left( \sum_i w_{ip}^n \tilde{v}_{i}^{n+1}(x_i^n - x_p^n)^T \right) : \epsilon
\]

\[
= \sum_p x_p^{n+1} \times m_p v_p^{n+1} + \sum_p \sum_i w_{ip}^n (x_i^n - x_p^n) \times \tilde{v}_{i}^{n+1}
\]

\[
= \sum_p x_p^{n+1} \times m_p v_p^{n+1} - \sum_p \sum_i w_{ip}^n x_i^n \times \tilde{v}_{i}^{n+1} + \sum_i \left( \sum_p w_{ip}^n \tilde{v}_{i}^{n+1} \right) x_i^n \times \tilde{v}_{i}^{n+1}
\]

\[
= \sum_p x_p^{n+1} \times m_p v_p^{n+1} - \sum_p \sum_i w_{ip}^n x_i^n \times \tilde{v}_{i}^{n+1} + \sum_i \left( \sum_p w_{ip}^n \tilde{v}_{i}^{n+1} \right) x_i^n \times \tilde{v}_{i}^{n+1}
\]

\[
= \sum_p x_p^{n+1} \times m_p v_p^{n+1} - \sum_p \sum_i w_{ip}^n x_i^n \times \tilde{v}_{i}^{n+1} + \sum_i \left( \sum_p w_{ip}^n \tilde{v}_{i}^{n+1} \right) x_i^n \times \tilde{v}_{i}^{n+1}
\]

\[
= \sum_p x_p^{n+1} \times m_p v_p^{n+1} - \sum_p \sum_i w_{ip}^n x_i^n \times \tilde{v}_{i}^{n+1} + \sum_i \left( \sum_p w_{ip}^n \tilde{v}_{i}^{n+1} \right) x_i^n \times \tilde{v}_{i}^{n+1}
\]

\[
= \sum_p \Delta t v_p^{n+1} \times m_p v_p^{n+1} + L_{\text{tot}}^{G,n+1}
\]

\[
= L_{\text{tot}}^{G,n+1}
\]